The Tetrahexes Puzzle

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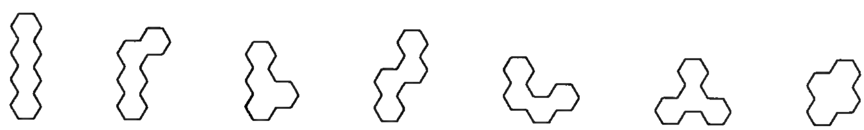
During the celebration of the 40-th anniversary of the Mathematical Centre on February 11, 1986, the participants received a small puzzle consisting of the following seven pieces, sawn from perspex:



These pieces represent the seven different ways to join four congruent hexagons.[1]

In the June 1967 issue of *Scientific American*, MARTIN GARDNER devoted a part of his famous column to this puzzle. He proposed to call the pieces 'polyhexes' after DAVID KLARNER, who was one of the first to investigate them (pieces consisting of four hexagons are called tetrahexes, pieces consisting of five hexagons are called pentahexes, and so on). The number of pentahexes is 22; computer programs have counted 82 hexahexes, 333 heptahexes and 1448 octahexes. No formula is known to determine the number of n-hexes for a given value of n. Together with the puzzle, 15 figures were provided as exercise material (the exercise being to form each figure with the seven tetrahexes).

1. A limited number of copies of this puzzle is available to visitors of the CWI who sign the guest book.

Below, we give 49 figures including 8 out of these 15. All of them (except the first one, see below) can be composed, in at least one way, by the seven different tetrahexes. The readers are invited to find solutions and, if possible, to find them all!

The first figure, the triangle, is to many the most appealing one which consists of 28 hexagons. Unfortunately, it has no solution with the seven different pieces, as will be shown below. Figures 2-35 represent all the solvable convex figures (by convex we mean that the curve which connects the midpoints of the outer hexagons is convex) with at most one convex hole and at least one symmetry (i.e. mirror or rotation symmetry). Although the hexagon has a 6-fold rotational symmetry, we found only one solvable pattern with 3-fold rotational symmetry (Figure 36). Figures 37-49 are solvable 'rectangular-like' figures with at most one hole and at least one symmetry. There may be others. For the triangle, DAVID KLARNER showed that it has no solution. As far as we know, Klarner's proof has not been published. Here, we will give a proof which we suspect to be similar to Klarner's proof.

PROOF THAT THE TRIANGLE CAN NOT BE COMPOSED OF THE SEVEN TETRAHEXES.

The positions of the triangle are numbered 1 - 28 as follows.

```
                  1
                2   3
              4   5   6
            7   8   9   10
         11  12  13  14  15
       16  17  18  19  20  21
     22  23  24  25  26  27  28
```

As Klarner did, we start by observing that the propeller can only be placed in a limited number of positions, viz., in positions 4 8 9 12, 5 8 7 13 and 8 13 14 18 (apart from mirror-image forms). When the propeller is in position 4 8 9 12, the bee should be in position 1 2 3 5. Next, there are three possible locations for the arch; and so on. The complete proof is given in a tree below.

Some abbreviations are being used: the different pieces are identified by their first two characters. An 'x' after a piece indicates that this can not be placed, or it splits the figure in parts with numbers of free hexagons which are not a multiple of four. A 'd' after a piece means that it should be placed somewhere in the figure, but has been placed somewhere else earlier.

```
pr 4 8 9 12
  be 1 2 3 5
    ar 6 10 13 14
      wa 15 20 19 25
        pi 21 28 27 26
          ba 7 11 16 22
            wo x
      wa 18 19 26 27
        pi 20 15 21 28
          ba 7 11 16 22
            wo x
          ba 22 23 24 25
            wo x
    ar 7 11 17 18
      pi 16 22 23 24
        wa 6 10 14 20
          ba x
        wa 25 19 20 15
          ba x
    ar 18 13 14 20
      ba 6 10 15 21
        wa x
```

```
pr 8 13 14 18
  ar 5 4 7 12
    pi 2 1 3 6
      wa 19 20 27 18
         wo 9 10 15 21
           be x
  ar 7 4 5 9
    pi 2 1 3 6
      wa x
```

```
pr 7 8 5 13
  pi 3 1 2 4
    ar 9 14 19 18
      ba 6 10 15 21
        pi d
      wo 6 10 15 20
        pi d
    ar 12 18 24 23
      pi d
    ar 12 18 19 14
      pi d
    ar 12 17 24 25
      pi d
    ar 16 11 12 18
      pi d
    ar 11 12 18 24
      be 17 16 23 22
        wa 6 9 14 19
          wo x
        wa 14 20 21 28
          ba x
        wa 25 19 20 15
          ba x
```

The reader is invited to find a shorter proof (than this 54-lines one).

Of course, our method also finds solutions, if they exist. For example, for figure 38, one rather quickly finds two solutions as follows (the complete tree is very long and yields 21 more solutions).

The positions are numbered as follows:

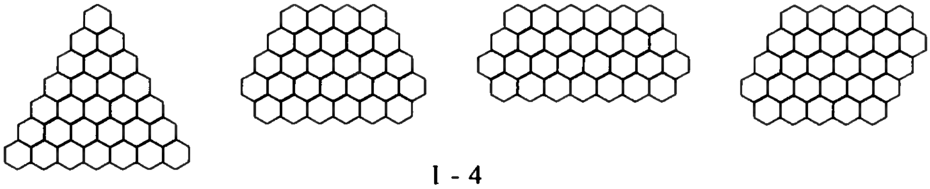
```
  1    2    3    4
5    6    7    8
  9   10   11   12
13   14   15   16
  17   18   19   20
21   22   23   24
  25   26   27   28
```

The solution tree then reads:

```
pr 3 7 6 11
  wa 4 8 12 16
    ar 2 1 5 9
      ba 17 18 19 20
        pi 10 15 14 13
          be 21 22 25 26
            wo x
          be 23 24 27 28
            wo x
      ba 21 22 23 24
        ba d
      ba 25 26 27 28
        be 10 14 15 18
          wo x
        be 18 17 22 21
          be d
        be 20 19 24 23
          wo 21 22 18 15
            pi 13 17 14 10  first solution
          wo 10 15 18 22
            pi 13 14 17 21  second solution
        be 13 14 17 18
          pi x
```

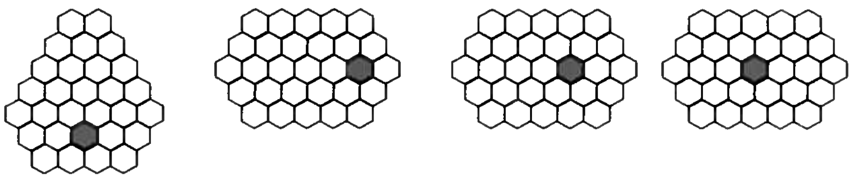
Of course, this may be programmed very efficiently in a language which allows for recursiveness. Some manual exercises with the above trees will certainly help to improve the performance of such programs.

FIGURES 1-49

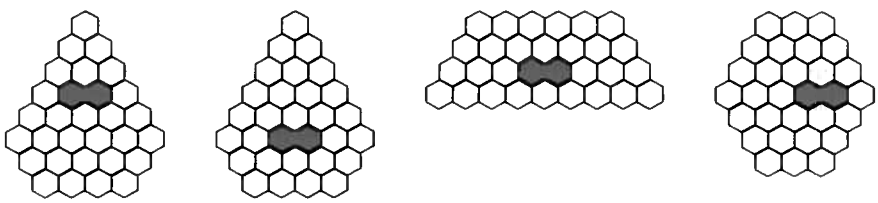


1 - 4

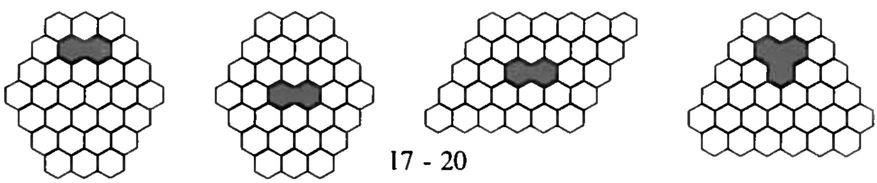
5 - 8



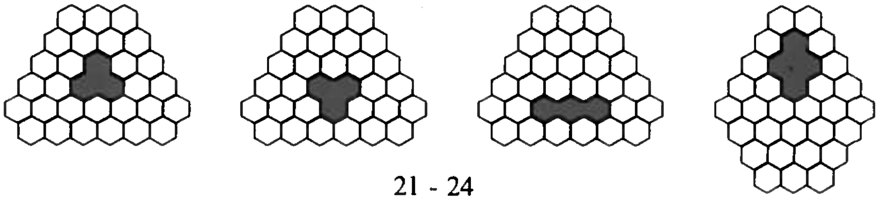
9 - 12



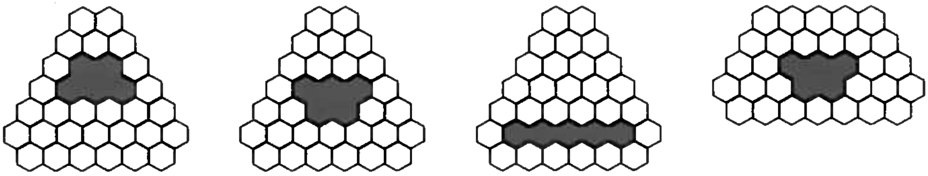
13 - 16



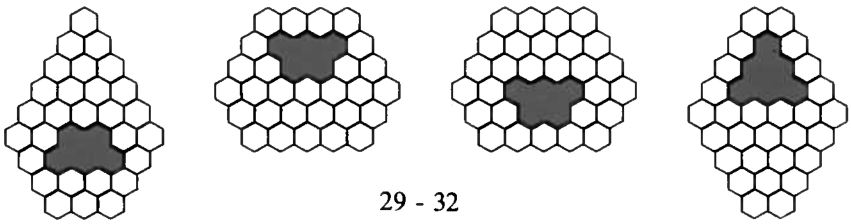
17 - 20



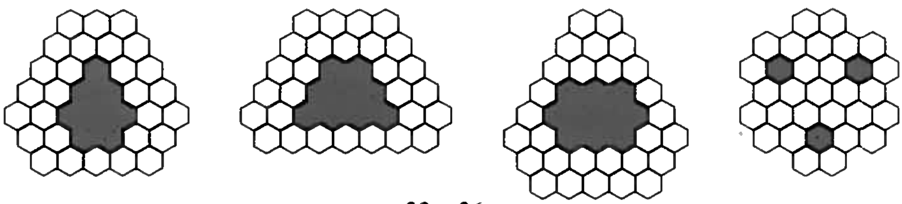
21 - 24



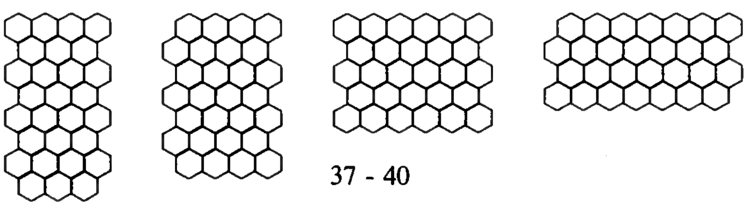
25 - 28



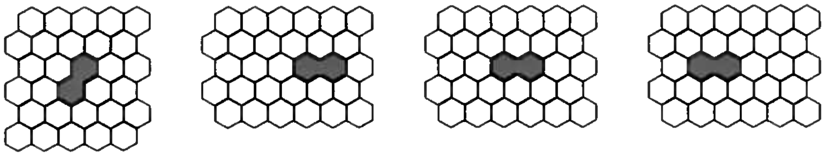
29 - 32



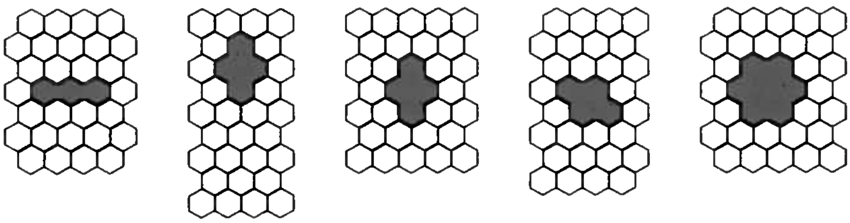
33 - 36



37 - 40



41 - 44



45 - 49